FACTORS OF HYPERCONTRACTIONS

MONOJIT BHATTACHARJEE AND B. KRISHNA DAS

Abstract. In this article, we study a class of contractive factors of m-hypercontractions for $m \in \mathbb{N}$. We find a characterization of such factors and it is achieved by finding explicit dilations of these factors on certain weighted Bergman spaces. This is a generalization of the work done in [13].

1. Introduction

The structure of a commuting n-tuple of isometries ($n \geq 2$) is complicated compared to that of a single isometry as an isometry always decomposes into a shift and a unitary due to von Neumann and Wold (cf. [19]). Not much is known except for the BCL representation for n-tuples of commuting isometries with product being a pure isometry (see [5, 6, 7, 8, 11, 16, 17] and references therein), that is for commuting isometries $(V_1, \dots, V_n)$ on $\mathcal{H}$ with

$$\cap_{k\geq 0} V_1^k V_2^k \cdots V_n^k \mathcal{H} = \{0\}.$$

The structure theorem of such isometries also reveals all possible isometric factors of a pure isometry (see [8] for more details). Following this, the analysis of finding factors has been extended further to the case of contractions in [13]. A characterization of contractive factors of a pure contraction is obtained, by Sarkar, Sarkar and the second author of this atricle, in [13] and subsequently in [21] for general contractions. It is also worth mentioning here that the key to obtaining such a characterization is an explicit Ando type dilation result which is motivated by a recent technique of explicit dilations of commuting contractions [12]. It is then natural to ask the following question: How to characterize contractive factors of m-hypercontractions? In this article, we answer this question and obtain a complete description for a class of contractive factors of m-hypercontractions. Our characterization for contractive factors of m-hypercontractions yields a similar characterization for a class of contractive factors of subnormal operators. To describe these results succinctly, we develop some background material next.

For a Hilbert space $\mathcal{E}$ and $n \in \mathbb{N}$, the $\mathcal{E}$-valued weighted Bergman space over the unit disc $\mathbb{D}$, denoted by $A_n^2(\mathcal{E})$, is defined as

$$A_n^2(\mathcal{E}) = \{f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k, \|f\|_n^2 = \sum_{k=0}^{\infty} (w_{n,k})^{-1} \|\hat{f}(k)\|_{\mathcal{E}}^2 < \infty\},$$

2010 *Mathematics Subject Classification.* 47A13, 47A20, 47A45, 47A56, 46E22, 47B32, 32A36, 47B20.

Key words and phrases. Hypercontraction, Bergman space over the unit disc, commuting contractions, bounded analytic functions.

where the sequence of weights $\{w_{n,k}\}_{k\geq 0}$ is given by

$$(1-x)^{-n} = \sum_{k=0}^{\infty} w_{n,k} x^k \quad (|x|<1).$$

The space $A^2_n(\mathcal{E})$ is also a reproducing kernel Hilbert space with kernel

$$K_n(z,w) = (1-z\bar{w})^{-n} I_{\mathcal{E}} \quad (z,w \in \mathbb{D}).$$

For the base case $n=1$, the space $A^2_1(\mathcal{E})$ is the Hardy space over the unit disc which we denote by $H^2_{\mathcal{E}}(\mathbb{D})$ and we denote the corresponding kernel, known as the Szegö kernel, by

$$\mathbb{S}(z,w) = (1-z\bar{w})^{-1} I_{\mathcal{E}} \quad (z,w \in \mathbb{D}).$$

If $\mathcal{E} = \mathbb{C}$, then we denote simply by A^2_n the $\mathbb{C}$-valued weighted Bergman space over the unit disc. Using Bergman kernels Agler, in his seminal paper [2], introduced the notion of m-hypercontractions ($m \in \mathbb{N}$) as follows. A bounded linear operator T on $\mathcal{H}$ is an m-hypercontraction if

$$K_n^{-1}(T,T^*) = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} T^k T^{*k} \geq 0,$$

for $n = 1, m$. In addition, if $T^{*n} \to 0$ in the strong operator topology then T is said to be a pure m-hypercontraction. It is important to note that the positivity $K_n^{-1}(T,T^*) \geq 0$ for $n = 1, m$ also implies all the intermediate positivity, that is $K_n^{-1}(T,T^*) \geq 0$ for all $n = 1, \ldots, m$ ([18]). This shows that if T is an m-hypercontraction then it is also an n-hypercontraction for $n = 1, \ldots, m$. The *defect operators* and the *defect spaces* of an m-hypercontraction T on $\mathcal{H}$ are defined by

$$D_{n,T} = \left(K_n^{-1}(T,T^*)\right)^{\frac{1}{2}} \text{ and } \mathcal{D}_{n,T} = \overline{\text{ran}} D_{n,T} \quad (1 \leq n \leq m) \tag{1.1}$$

respectively. The Bergman shift M_z on $A^2_m(\mathcal{E})$, defined by

$$\big(M_z f\big)(w) = w f(w) \quad (f \in A^2_m(\mathcal{E}), w \in \mathbb{D}),$$

is a pure m-hypercontraction. In fact, by [2], Bergman shifts are models for pure m-hypercontractions. To be more precise, Agler proves the following characterization result.

Theorem 1.1. (cf. [2]) *If T is an m-hypercontraction on a Hilbert space $\mathcal{H}$ then*

$$T \cong P_{\mathcal{Q}}(M_z \oplus W)|_{\mathcal{Q}},$$

where W is a unitary on a Hilbert space $\mathcal{R}$, $\mathcal{Q}$ is a $(M_z^* \oplus W^*)$-invariant subspace of $A^2_m(\mathcal{D}_{m,T}) \oplus \mathcal{R}$ and $\mathcal{D}_{m,T}$ is the defect space of T as in (1.1). *In particular, if T is pure then*

$$T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}},$$

where $\mathcal{Q}$ is a M_z^*-invariant subspace of $A^2_m(\mathcal{D}_{m,T})$.

There are now several different approach to this result and to its multivariable generalization for different domains in $\mathbb{C}^n$ (see [1], [3], [9], [10] , [18] and [20]).
Coming back to the context of this article, we denote by $\mathcal{F}_m(\mathcal{H})$ the class of contractive factors of m-hypercontractions on a Hilbert space $\mathcal{H}$ which we characterize in this paper. The class is defined as follows.

Definition 1.2. *For $m \in \mathbb{N}$ and a Hilbert space $\mathcal{H}$, a pair of commuting contractions (T_1, T_2) on $\mathcal{H}$ is said to be an element of $\mathcal{F}_m(\mathcal{H})$ if for $i = 1, 2$*

$$K_{m-1}^{-1}(T, T^*) - T_i K_{m-1}^{-1}(T, T^*) T_i^* \geq 0, \tag{1.2}$$

where $T = T_1 T_2$ and $K_0(T, T^*) = I_{\mathcal{H}}$.

The positivity condition in the above definition is equivalent to the Szegö positivity of the commuting m-tuple

$$\mathcal{T}_i = (\underbrace{T, \dots, T}_{(m-1)-\text{times}}, T_i)$$

for $i = 1, 2$. Here an n-tuple of commuting contraction $\mathcal{T} = (T_1, \dots, T_n)$ satisfies Szegö positivity if

$$\mathbb{S}_n^{-1}(\mathcal{T}, \mathcal{T}^*) = \sum_{F \subset \{1, \dots, n\}} (-1)^{|F|} \mathcal{T}_F \mathcal{T}_F^* \geq 0,$$

where for $F \subset \{1, \dots, n\}$, $\mathcal{T}_F = \prod_{i \in F} T_i$. If $m = 1$, it follows that $\mathcal{F}_1(\mathcal{H})$ is the class of all commuting contractive operator pairs on $\mathcal{H}$. For $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$, we show that the product $T = T_1 T_2$ is an m-hypercontraction on $\mathcal{H}$. In other words, for any $m \in \mathbb{N}$, $\mathcal{F}_m(\mathcal{H})$ contains contractive factors of m-hypercontractions on $\mathcal{H}$. In particular, the positivity condition (1.2) is a sufficient condition for the product of a pair of commuting contractions (T_1, T_2) on $\mathcal{H}$ to be an m-hypercontraction. This sufficient condition is not necessary (see the counterexample obtain in Section 6). The goal of this article is to present a complete description of the class of contractive factors $\mathcal{F}_m(\mathcal{H})$ of m-hypercontractions. One such explicit descriptions we obtain is as follows. For a Hilbert space $\mathcal{E}$, an operator valued analytic function $\Phi : \mathbb{D} \to \mathcal{B}(\mathcal{E})$ is a $\mathcal{B}(\mathcal{E})$-valued Schur function on $\mathbb{D}$ if

$$\sup_{z \in \mathbb{D}} \|\Phi(z)\| \leq 1.$$

Theorem. *If T is a m-hypercontraction on a Hilbert space $\mathcal{H}$, then the following are equivalent:*

(i) *$T = T_1 T_2$ for some $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$;*

(ii) *there exist a pair of commuting unitaries (W_1, W_2) on a Hilbert space $\mathcal{R}$ with $W = W_1 W_2$ and a pair of $\mathcal{B}(\mathcal{E})$-valued Schur functions on $\mathbb{D}$*

$$\Phi(z) = (P + zP^{\perp})U^*, \text{ and } \Psi(z) = U(P^{\perp} + zP), \quad (z \in \mathbb{D})$$

corresponding to a triple $(\mathcal{E}, U, P)$ consisting of a Hilbert space $\mathcal{E}$, a unitary U on $\mathcal{E}$ and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that $\mathcal{Q}$ is a joint $(M_z^* \oplus W^*, M_\Phi^* \oplus W_1^*, M_\Psi^* \oplus W_2^*)$-invariant subspace of $A_m^2(\mathcal{E}) \oplus \mathcal{R}$ and

$$T_1 \cong P_{\mathcal{Q}}(M_\Phi \oplus W_1)|_{\mathcal{Q}}, \; T_2 \cong P_{\mathcal{Q}}(M_\Psi \oplus W_2)|_{\mathcal{Q}}, \; T \cong P_{\mathcal{Q}}(M_z \oplus W)|_{\mathcal{Q}}.$$

Moreover, if T is a pure m-hypercontraction, then the Hilbert space $\mathcal{R} = \{0\}$.

This theorem is proved in Section 5 as Theorem 5.1 and is obtained by finding a suitable and explicit dilation of commuting contractive operator triple (T_1, T_2, T_1T_2), $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$, on some weighted Bergman space. At the same time, the explicit dilations of triples relies on a Douglus type dilations of m-hypercontractions and a commutant lifting technique found in [13]. The above factorization result, in turn, also provides a similar factorization result for subnormal operators and, for $m = 1$, it recovers the characterization of contractive factors of contractions obtained in [13] and [21].

The plan of the paper is as follows. Section 2 contains Douglus type dilations for m-hypercontractions. We study different properties of $\mathcal{F}_m(\mathcal{H})$ in Section 3. In Section 4, we find suitable explicit dilations for the class of factors in $\mathcal{F}_m(\mathcal{H})$. This is then used to obtain several factorization results in Section 5. In the last section, we find examples of factors of m-hypercontractions on $\mathcal{H}$ which are not elements of $\mathcal{F}_m(\mathcal{H})$.

2. Douglas type dilations for hypercontractions

Ever since Sz.-Nagy and Foias discovered unitary dilations of contractions, several explicit constructions of such unitary dilations have been obtained. One such construction is due to Douglas [14]. Here we carry out a similar construction for m-hypercontrcations and obtain Douglas type dilations for m-hypercontractions. Our explicit construction of Douglas type dilations for m-hypercontractions seems to be new and it is used to obtain dilations of factors of m-hypercontractions.

Recall that a linear operator T on $\mathcal{H}$ is a m-hypercontraction if for $n = 1, \dots, m$,

$$K_n^{-1}(T, T^*) = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} T^k T^{*k} \geq 0.$$

Also for $n = 1, \dots, m$, n-th order defect operator and defect space are

$$D_{n,T} = K_n^{-1}(T, T^*)^{1/2} \text{ and } \mathcal{D}_{n,T} = \overline{\text{ran}} D_{n,T},$$

respectively. The sequence of weights $\{w_{n,k}\}_{k=0}^{\infty}$ given by

$$(1 - x)^{-n} = \sum_{k=0}^{\infty} w_{n,k} x^k, \qquad (|x| < 1, n \in \mathbb{N} \cup \{0\})$$

plays a crucial role in what follows and we invoke a lemma from [2] which describes certain relationship of these weights for different values of n.

Lemma 2.1 (cf [2]). *Let $\{w_{n,k}\}_{k \geq 0, n \geq 0}$ be as above. Then for all $n, k \geq 1$,*

$$w_{n,k} - w_{n,k-1} = w_{n-1,k}.$$

For a fixed n with $1 \leq n \leq m$, consider an orthonormal basis $\{\psi_{n,k}(z) = \sqrt{w_{n,k}} z^k\}_{k=0}^{\infty}$ for the weighted Bergman space A_n^2. Then the kernel function of A_n^2 is given by

$$K_n(z, w) = (1 - z\bar{w})^{-n} = \sum_{k=0}^{\infty} \overline{\psi_{n,k}(w)} \psi_{n,k}(z) \quad (z, w \in \mathbb{D}).$$

We set, for $r \geq 0$,

$$f_r^{(n)}(z,w) := \sum_{k=r}^{\infty} \psi_{n,k}(z) K_n^{-1}(z,w) \overline{\psi_{n,k}(w)} \quad (z,w \in \mathbb{D}).$$

Then it can be easily seen that $f_0^{(n)} \equiv 1$ and

$$f_j^{(n)}(z,w) = 1 - \sum_{k=0}^{j-1} \psi_{n,k}(z) K_n^{-1}(z,w) \overline{\psi_{n,k}(w)}, \ (j \geq 1)$$

and consequently, $f_r^{(n)}$ is a polynomial for all $r \geq 0$. As a result, using polynomial calculus, we define

$$f_r^{(n)}(T,T^*) := 1 - \sum_{k=0}^{r-1} w_{n,k} T^k K_n^{-1}(T,T^*) T^{*k}, \quad (r \geq 0, 1 \leq n \leq m)$$

for any m-hypercontraction T on $\mathcal{H}$. These operators are used to study the canonical dilation map $\Pi_{m,T} : \mathcal{H} \to A_m^2(\mathcal{D}_T)$ defined by

$$(\Pi_{m,T} h)(z) = D_{m,T}(I_{\mathcal{H}} - zT^*)^{-m} h, \qquad (h \in \mathcal{H}, z \in \mathbb{D}) \tag{2.1}$$

corresponding to an m-hypercontraction T on $\mathcal{H}$. The next proposition shows that the canonical dilation map $\Pi_{m,T}$ is a contraction and it is analogous to Proposition 10 in [3] for the case when T is a pure m-hypercontraction.

Proposition 2.2. *In the above setting, we have the following:*

(i) *For any n with $1 \leq n \leq m$, the sequence $\{f_r^{(n)}(T,T^*)\}_{r=0}^{\infty}$ is a decreasing sequence of positive operators.*

(ii) $\|\Pi_{m,T} h\|^2 = \|h\|^2 - \lim_{r\to\infty} \langle f_r^{(m)}(T,T^*)h, h\rangle \quad (h \in \mathcal{H})$.

Proof. It is clear from the definition that $\{f_r^{(n)}(T,T^*)\}_{r=0}^{\infty}$ is a decreasing sequence for $n = 1, \ldots, m$. For the positivity, it follows from Lemma 2.1 and the discussion succeeding it that for all $r \geq 0$ and $1 \leq n \leq m$,

$$\begin{aligned} f_r^{(n)}(T,T^*) &= 1 - \sum_{k=0}^{r-1} w_{n,k} T^k K_n^{-1}(T,T^*) T^{*k} \\ &= 1 - \sum_{k=0}^{r-1} w_{n,k} T^k \Big(K_{n-1}^{-1}(T,T^*) - T K_{n-1}^{-1}(T,T^*) T^* \Big) T^{*k} \\ &= 1 - w_{n,0} K_{n-1}^{-1}(T,T^*) - \sum_{k=1}^{r-1} (w_{n,k} - w_{n,k-1}) T^k K_{n-1}^{-1}(T,T^*) T^{*k} \\ &\qquad + w_{n,r-1} T^r K_{n-1}^{-1}(T,T^*) T^{*r} \\ &= f_r^{(n-1)}(T,T^*) + w_{n,r-1} T^r K_{n-1}^{-1}(T,T^*) T^{*r}. \end{aligned} \tag{2.2}$$

Since $w_{n,r-1}T^rK_{n-1}^{-1}(T,T^*)T^{*r} \geq 0$, we conclude that $f_r^{(n)}(T,T^*) \geq f_r^{(n-1)}(T,T^*)$ for all $r \geq 0$ and for $n = 1, \ldots, m$. As a result, we also have

$$f_r^{(n)}(T,T^*) \geq f_r^{(n-1)}(T,T^*) \geq \cdots \geq f_r^{(1)}(T,T^*) = T^rT^{*r} \geq 0.$$

This proves that $\{f_r^{(n)}(T,T^*)\}_{r=0}^{\infty}$ is a decreasing sequence of positive operators. The proof of (ii) is verbatim with the proof of Proposition 10 in [3]. ■

By the above result, we denote the strong operator limit of the sequence $\{f_r^{(n)}(T,T^*)\}_{r=0}^{\infty}$ and its range as

$$Q_{n,T}^2 := \text{SOT} - \lim_{r\to\infty} f_r^{(n)}(T,T^*) \quad \mathcal{Q}_{n,T} = \overline{ran} Q_{n,T} \quad (1 \leq n \leq m). \tag{2.3}$$

It should be noted that if T is a pure m-hypercontraction then

$$\text{SOT} - \lim_{r\to\infty} f_r^{(m)}(T,T^*) = \text{SOT} - \lim_{r\to\infty} f_r^{(m-1)}(T,T^*) = \cdots = \text{SOT} - \lim_{r\to\infty} f_r^{(1)}(T,T^*) = 0.$$

This can be derived from the identity (2.2) and from the fact that $w_{n,r-1}T^rK_{n-1}^{-1}(T,T^*)T^{*r} \to 0$ in the strong operator topology (see Lemma 2.11 in [2]). Thus the canonical dilation map $\Pi_{m,T}$ is an isometry if and only if T is a pure m-hypercontraction. The intertwining property of $\Pi_{m,T}$, that is $\Pi_{m,T}T^* = M_z^*\Pi_{m,T}$ where M_z is the shift on $A_m^2(\mathcal{D}_{m,T})$, is evident from the definition of $\Pi_{m,T}$.

Before we present the main theorem of this section, we recall a well-known factorization result due to Douglas.

Lemma 2.3. (cf. [15]) *Let A and B be two bounded linear operators on a Hilbert space $\mathcal{H}$. Then there exists a contraction C on $\mathcal{H}$ such that $A = BC$ if and only if*

$$AA^* \leq BB^*.$$

The explicit construction of Douglas type dilation for m-hypercontractions is given next.

Theorem 2.4. *If $T \in \mathcal{B}(\mathcal{H})$ is an m-hypercontraction, then there exist a Hilbert space $\mathcal{R}$, an isometry $\Pi_T : \mathcal{H} \to A_m^2(\mathcal{D}_{m,T}) \oplus \mathcal{R}$ and a unitary W on $\mathcal{R}$ such that*

$$\Pi_T T^* = (M_z^* \oplus W^*)\Pi_T.$$

In particular,

$$T \cong P_{\mathcal{Q}}(M_z \oplus W)|_{\mathcal{Q}},$$

where $\mathcal{Q} = ran\Pi_T$ is the $(M_z \oplus W)^*$-invariant subspace of $A^2(\mathcal{D}_{m,T}) \oplus \mathcal{R}$.

Proof. Let $Q_{n,T}$ be the positive operator as in (2.3) for all $1 \leq n \leq m$. By induction on n, we prove that

$$TQ_{n,T}^2T^* = Q_{n,T}^2 \ (n = 1, \ldots, m).$$

It is easy to see that it holds for $n = 1$. Then we assume that the identity holds for some n with $1 \leq n < m$. Thus by the assumption $f_{r+1}^{(n)}(T,T^*) - Tf_{r+1}^{(n)}(T,T^*)T^* \to 0$ in the strong operator topology as $r \to \infty$. Now,

$$
\begin{aligned}
& f_{r+1}^{(n+1)}(T,T^*) - Tf_r^{(n+1)}(T,T^*)T^* \\
&= I - TT^* - K_{n+1}^{-1}(T,T^*) + \sum_{k=0}^{r-1}(w_{n+1,k} - w_{n+1,k+1})T^{k+1}K_{n+1}^{-1}(T,T^*)T^{*(k+1)} \\
&= I - TT^* - K_{n+1}^{-1}(T,T^*) - \sum_{k=0}^{r-1} w_{n,k+1}T^{k+1}K_{n+1}^{-1}(T,T^*)T^{*(k+1)} \\
&= I - TT^* - (K_n^{-1}(T,T^*) - TK_n^{-1}(T,T^*)T^*) \\
&\qquad\qquad\qquad - \sum_{k=0}^{r-1} w_{n,k+1}T^{k+1}\big(K_n^{-1}(T,T^*) - TK_n^{-1}(T,T^*)T^*\big)T^{*(k+1)} \\
&= \big(I - \sum_{k=0}^{r} w_{n,k}T^kK_n^{-1}(T,T^*)T^{*k}\big) - \big(TT^* - \sum_{k=0}^{r} w_{n,k}T^{k+1}K_n^{-1}(T,T^*)T^{*(k+1)}\big) \\
&= f_{r+1}^{(n)}(T,T^*) - Tf_{r+1}^{(n)}(T,T^*)T^*.
\end{aligned}
$$

Consequently by the induction hypothesis, $f_{r+1}^{(n+1)}(T,T^*) - Tf_r^{(n+1)}(T,T^*)T^* \to 0$ in the strong operator topology as $r \to \infty$. This in turn implies that

$$TQ_{n+1,T}^2T^* = Q_{n+1,T}^2.$$

Thus we have proved that $TQ_{n,T}^2T^* = Q_{n,T}^2$ for $n = 1, \ldots, m$. In particular, since $TQ_{m,T}^2T^* = Q_{m,T}^2$, by Lemma 2.3, there exists an isometry X^* on $\mathcal{Q}_{m,T}$ such that

$$X^*Q_{m,T} = Q_{m,T}T^*. \tag{2.4}$$

Let W^* on $\mathcal{R} \supseteq \mathcal{Q}_{m,T}$ be the minimal unitary extension of X^* ([19]). Then, by Proposition 2.2, the map $\Pi_T : \mathcal{H} \to A_m^2(\mathcal{D}_{m,T}) \oplus \mathcal{R}$ defined by

$$\Pi_T h = (\Pi_{m,T}h, Q_{m,T}h), \quad (h \in \mathcal{H})$$

is an isometry and it also satisfies

$$\Pi_T T^* = (M_z \oplus W)^*\Pi_T.$$

Here the intertwining property follows from (2.4). Therefore, $\mathcal{Q} = \mathrm{ran}\Pi_T$ is an $(M_z \oplus W)^*$-invariant subspace of $A_m^2(\mathcal{D}_{m,T}) \oplus \mathcal{R}$ and

$$T \cong P_{\mathcal{Q}}(M_z \oplus W)|_{\mathcal{Q}}.$$

This completes the proof. ■

3. The class $\mathcal{F}_m(\mathcal{H})$

The class of contractive factors $\mathcal{F}_m(\mathcal{H})$ and its basic properties are studied in this section. To begin with we recall the definition of the class $\mathcal{F}_m(\mathcal{H})$. A commuting pair of contractions (T_1, T_2) on $\mathcal{H}$ is in $\mathcal{F}_m(\mathcal{H})$ if $K_{m-1}^{-1}(T,T^*) - T_iK_{m-1}^{-1}(T,T^*)T_i^* \geq 0$ for $i = 1, 2$, where $T = T_1T_2$.

For $(T_1,T_2)\in\mathcal{F}_m(\mathcal{H})$ with $T=T_1T_2$, we fix the following notations for the rest of the article:

$$D^2_{n,T,T_i}=K^{-1}_{n-1}(T,T^*)-T_iK^{-1}_{n-1}(T,T^*)T_i^* \text{ and } \mathcal{D}_{n,T,T_i}=\overline{\mathrm{ran}}D^2_{n,T,T_i}\quad (n\in\mathbb{N}, i=1,2). \tag{3.5}$$

With the above notation, we have the following useful identity

$$\begin{aligned}
&D^2_{n,T,T_i}-TD^2_{n,T,T_i}T^*\\
&=K^{-1}_{n-1}(T,T^*)-TK^{-1}_{n-1}(T,T^*)T^*-T_i\big(K^{-1}_{n-1}(T,T^*)-TK^{-1}_{n-1}(T,T^*)T^*\big)T_i^*\\
&=K^{-1}_{n}(T,T^*)-T_iK^{-1}_{n}(T,T^*)T_i^*\\
&=D^2_{n+1,T,T_i},
\end{aligned}\tag{3.6}$$

for all $n\geq 0$. Next we show an intermediate positivity type result.

Lemma 3.1. *If $(T_1,T_2)\in\mathcal{F}_m(\mathcal{H})$ then $(T_1,T_2)\in\mathcal{F}_n(\mathcal{H})$ for all $1\leq n\leq m$.*

Proof. It is enough to show that $D^2_{n,T,T_i}\geq 0$ for $n=1,\dots,m$ and for $i=1,2$. We only consider the case $i=1$ as it is symmetrical for $i=2$. By the hypothesis $D^2_{m,T,T_1}\geq 0$ and $D^2_{1,T,T_1}\geq 0$. To show $D^2_{(m-1),T,T_1}\geq 0$, we assume that $m\geq 2$ and consider the sequence $\{a_r\}_{r=0}^\infty$, corresponding to a fixed $h\in\mathcal{H}$, as

$$a_r=\langle T^rD^2_{(m-1),T,T_1}T^{*r}h,h\rangle\quad (r\geq 0).$$

Then for any $r\geq 0$, using (3.6), we have

$$\begin{aligned}
a_r-a_{r+1}&=\langle T^r(D^2_{(m-1),T,T_1}-TD^2_{(m-1),T,T_1}T^*)T^{*r}h,h\rangle\\
&=\langle T^rD^2_{m,T,T_1}T^{*r}h,h\rangle\geq 0.
\end{aligned}$$

Thus $\{a_r\}_{r=0}^\infty$ is a decreasing sequence. Also since

$$\begin{aligned}
\Big|\sum_{r=0}^N a_r\Big|&=\Big|\Big\langle\sum_{r=0}^N T^r(D^2_{(m-2),T,T_1}-TD^2_{(m-2),T,T_1}T^*)T^{*r}h,h\Big\rangle\Big|\\
&=\Big|\Big\langle(D^2_{(m-2),T,T_1}-T^{N+1}D^2_{(m-2),T,T_1}T^{*(N+1)})h,h\Big\rangle\Big|\\
&\leq 2\|h\|^2\|D^2_{(m-2),T,T_1}\|,
\end{aligned}$$

$a_r\geq 0$ for all $r\geq 0$. In particular, it implies that $D^2_{(m-1),T,T_1}\geq 0$. Therefore, by induction on m, we have all the required positivity. This completes the proof. ■

Needless to say that the product of two commuting contractions is not an m-hypercontraction, in general. We find a sufficient condition for product of two commuting contractions to be an m-hypercontraction. The sufficient condition is simply that the pair of contractions on $\mathcal{H}$ should be an element of $\mathcal{F}_m(\mathcal{H})$. This is proved in the next lemma, which is in the same spirit as Lemma 3.1 in [4].

Lemma 3.2. *If $(T_1,T_2)\in\mathcal{F}_m(\mathcal{H})$, then T_1T_2 is a m-hypercontraction.*

Proof. Let $T = T_1T_2$. The proof is obvious for $m = 1$. For $m \geq 2$ note that

$$\begin{aligned}&K_m^{-1}(T, T^*)\\ &= K_{m-1}(T, T^*) - TK_{m-1}^{-1}(T, T^*)T^*\\ &= \big(K_{m-1}^{-1}(T, T^*) - T_1^*K_{m-1}^{-1}(T, T^*)T_1^*\big) + T_1\big(K_{m-1}^{-1}(T, T^*) - T_2^*K_{m-1}^{-1}(T, T^*)T_2^*\big)T_1^* \geq 0.\end{aligned}$$

This completes the proof. ∎

The converse of this lemma is not true (see the counterexample in the last section). This suggests that $\mathcal{F}_m(\mathcal{H})$ does not contain all the factors of m-hypercontractions. Before going further, we consider elementary examples of elements in $\mathcal{F}_m(\mathcal{H})$. These examples are based on a triple $(\mathcal{E}, U, P)$ consists of a Hilbert space $\mathcal{E}$, a unitary operator U on $\mathcal{E}$ and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$. For such a triple, the $\mathcal{B}(\mathcal{E})$-valued analytic functions

$$\Phi(z) = (P + zP^{\perp})U^*, \text{ and } \Psi(z) = U(P^{\perp} + zP) \quad (z \in \mathbb{D})$$

are easily seen to be Schur functions on $\mathbb{D}$, that is Φ and Ψ are in the unit ball of the Banach algebra $H^{\infty}_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ consisting of bounded $\mathcal{B}(\mathcal{E})$-valued analytic functions on $\mathbb{D}$. It is also easy to see that

$$\Phi(z)\Psi(z) = \Psi(z)\Phi(z) = zI_{\mathcal{E}} \quad (z \in \mathbb{D}).$$

We refer to Φ, Ψ as *canonical pair of Schur functions* on $\mathbb{D}$ corresponding to the triple $(\mathcal{E}, U, P)$. We claim that the commuting pair of multiplication operators (M_{Φ}, M_{Ψ}) on $A^2_m(\mathcal{E})$ is an element of $\mathcal{F}_m(A^2_m(\mathcal{E}))$. Indeed, if $\mathcal{E}_1 = \text{ran}P$ and $\mathcal{E}_2 = \text{ran}P^{\perp}$ then $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. With respect to this decomposition of the co-efficient space $\mathcal{E}$, we have $A^2_m(\mathcal{E}) = A^2_m(\mathcal{E}_1) \oplus A^2_m(\mathcal{E}_2)$ and

$$\begin{aligned}&K_{m-1}(M_z, M_z^*) - M_{\Phi}K_{m-1}(M_z, M_z^*)M_{\Phi}^*\\ &= K_{m-1}(M_z, M_z^*) - \begin{bmatrix} I_{A^2_m(\mathcal{E}_1)} & 0\\ 0 & M_z \otimes I_{\mathcal{E}_2}\end{bmatrix} (I \otimes U^*)K_{m-1}(M_z, M_z^*)(I \otimes U) \begin{bmatrix} I_{A^2_m(\mathcal{E}_1)} & 0\\ 0 & M_z^* \otimes I_{\mathcal{E}_2}\end{bmatrix}\\ &= K_{m-1}(M_z, M_z^*) - \begin{bmatrix} I_{A^2_m(\mathcal{E}_1)} & 0\\ 0 & M_z \otimes I_{\mathcal{E}_2}\end{bmatrix} K_{m-1}(M_z, M_z^*) \begin{bmatrix} I_{A^2_m(\mathcal{E}_1)} & 0\\ 0 & M_z^* \otimes I_{\mathcal{E}_2}\end{bmatrix}\\ &= \begin{bmatrix} 0 & 0\\ 0 & K_m(M_z \otimes I_{\mathcal{E}_2}, M_z^* \otimes I_{\mathcal{E}_2})\end{bmatrix} \geq 0,\end{aligned}$$

as $M_z \otimes I_{\mathcal{E}_2}$ on $A^2_m(\mathcal{E}_2)$ is an m-hypercontraction. Similarly, we have

$$K_{m-1}(M_z, M_z^*) - M_{\Psi}K_{m-1}(M_z, M_z^*)M_{\Psi}^* \geq 0.$$

This proves the claim. In fact we will see below that any pair $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$ with T_1T_2 is pure dilates to such a pair (M_{Φ}, M_{Ψ}) on $A^2_m(\mathcal{E})$, and therefore these operator pairs serve as a model operator for a class of contractive factors of pure m-hypercontractions.

4. Dilation of factors

Our main concern is to propose a model for the class $\mathcal{F}_m(\mathcal{H})$ of contractive factors of m-hypercontractions. This is achieved by finding an explicit dilation of a triple of commuting contractions (T_1, T_2, T_1T_2) on some weighted Bergman space, where $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$. We say that an n-tuple of commuting contractions $(T_1, \dots, T_n)$ on $\mathcal{H}$ dilates to a commuting n-tuple of operators $(R_1, \dots, R_n)$ on $\mathcal{K}$ if there is an isometry $\Pi : \mathcal{H} \to \mathcal{K}$, such that

$$\Pi T_i^* = R_i^* \Pi \quad (i = 1, \dots, n).$$

The map Π is often refer as the dilation map.

We prove a lemma which will be the key to the dilations obtained in this section. This lemma is analogous to Theorem 2.1 in [13]. Let $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$. Since $T = T_1T_2$ is an m-hypercontraction, the canonical dilation map $\Pi_{m,T} : \mathcal{H} \to A^2_m(\mathcal{D}_{m,T})$ defined by

$$(\Pi_{m,T}h)(z) = D_{m,T}(I - zT^*)^{-m}h \quad (h \in \mathcal{H}, z \in \mathbb{D}),$$

satisfies $\Pi_{m,T}T^* = M_z^*\Pi_{m,T}$. If $V : \mathcal{D}_{m,T} \to \mathcal{E}$ is a isometry for some Hilbert space $\mathcal{E}$, then the map

$$\Pi_V := (I \otimes V)\Pi_{m,T} : \mathcal{H} \to A^2_m(\mathcal{E})$$

also intertwines with T^* and M_z^* on $A^2_m(\mathcal{E})$, that is $\Pi_V T^* = M_z^*\Pi_V$.

Lemma 4.1. *With the above notation, if $\mathcal{D}$ is a Hilbert space and if*

$$U_i = \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} : \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,T_i}) \to \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,T_i}) \quad (i = 1, 2)$$

is a unitary operator such that for all $h \in \mathcal{H}$,

$$U_i\big(VD_{m,T}h, 0_{\mathcal{D}}, D_{m,T,T_i}T^*h\big) = \big(VD_{m,T}T_i^*h, 0_{\mathcal{D}}, D_{m,T,T_i}h\big), \ (i = 1, 2)$$

then the $\mathcal{B}(\mathcal{E})$-valued Schur function $\Phi_i(z) = A_i^* + zC_i^*B_i^*$ ($z \in \mathbb{D}$), transfer function corresponding to the unitary U_i^*, satisfies

$$\Pi_V T_i^* = M^*_{\Phi_i}\Pi_V,$$

for $i = 1, 2$.

Proof. Since

$$\begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} \begin{bmatrix} VD_{m,T}h \\ (0_{\mathcal{D}}, D_{m,T,T_i}T^*h) \end{bmatrix} = \begin{bmatrix} VD_{m,T}T_i^*h \\ (0_{\mathcal{D}}, D_{m,T,T_i}h) \end{bmatrix}, \quad (h \in \mathcal{H}, i = 1, 2)$$

we have

$$A_iVD_{m,T}h + B_i(0_{\mathcal{D}}, D_{m,T,T_i}T^*h) = VD_{m,T}T_i^*h \text{ and } C_iVD_{m,T}h = (0_{\mathcal{D}}, D_{m,T,T_i}h)$$

for all $h \in \mathcal{H}$ and $i = 1, 2$. Simplifying further, we get

$$VD_{m,T}T_i^* = A_iVD_{m,T} + B_iC_iVD_{m,T}T^*$$

for $i = 1, 2$. Finally, if $n \geq 1$, $h \in \mathcal{H}$ and $\eta \in \mathcal{E}$, then

$$\begin{aligned} \langle M^*_{\Phi_i}\Pi_V h, z^n\eta\rangle &= \langle (I \otimes V)D_{m,T}(1 - zT^*)^{-m}h, (A_i^* + zC_i^*B_i^*)(z^n\eta)\rangle \\ &= \langle (A_iVD_{m,T} + B_iC_iVD_{m,T^*}T^*)T^{*n}h, \eta\rangle \\ &= \langle VD_{m,T}T_i^*(T^{*n}h), \eta\rangle \\ &= \langle \Pi_V T_i^*h, z^n\eta\rangle, \quad (i = 1, 2). \end{aligned}$$

Therefore, we get $\Pi_V T_i^* = M^*_{\Phi_i}\Pi_V$ for $i = 1, 2$. This completes the proof. ∎

Let $(T_1,T_2)\in\mathcal{F}_m(\mathcal{H})$ with $T=T_1T_2$. Then (see the proof of Lemma 3.2)

$$K_m^{-1}(T,T^*)=D^2_{m,T,T_1}+T_1D^2_{m,T,T_2}T_1^*=D^2_{m,T,T_2}+T_2D^2_{m,T,T_1}T_2^*,$$

implies that

$$\|D_{m,T,T_1}h\|^2+\|D_{m,T,T_2}T_1^*h\|^2=\|D_{m,T,T_2}h\|^2+\|D_{m,T,T_1}T_2^*h\|^2$$

for all $h\in\mathcal{H}$. This leads us to define isometries $V:\mathcal{D}_{m,T}\to\mathcal{D}_{m,T,T_1}\oplus\mathcal{D}_{m,T,T_2}$ and

$$U:\{D_{m,T,T_1}T_2^*h\oplus D_{m,T,T_2}h:h\in\mathcal{H}\}\to\{D_{m,T,T_1}h\oplus D_{m,T,T_2}T_1^*h:h\in\mathcal{H}\}$$

defined by

$$V(D_{m,T}h)=(D_{m,T,T_1}h,D_{m,T,T_2}T_1^*h)\quad(h\in\mathcal{H})\tag{4.7}$$

and

$$U\big(D_{m,T,T_1}T_2^*h,D_{m,T,T_2}h\big)=\big(D_{m,T,T_1}h,D_{m,T,T_2}T_1^*h\big),\quad(h\in\mathcal{H})\tag{4.8}$$

respectively. We are now ready to prove the explicit dilation result for the pure case.

THEOREM 4.2. *Let $(T_1,T_2)\in\mathcal{F}_m(\mathcal{H})$ be such that $T=T_1T_2$ is a pure contraction. Then there exist a triple $(\mathcal{E},U,P)$ consists of a Hilbert space $\mathcal{E}$, a unitary U and a projection P in $\mathcal{B}(\mathcal{E})$, and an isometry $\Pi:\mathcal{H}\to A^2_m(\mathcal{E})$ such that*

$$\Pi T_1^*=M_\Phi^*\Pi,\ \Pi T_2^*=M_\Psi^*\Pi,\ \text{and}\ \Pi T^*=M_z^*\Pi,$$

where Φ and Ψ are the $\mathcal{B}(\mathcal{E})$-valued canonical Schur functions on $\mathbb{D}$ corresponding to the triple $(\mathcal{E},U,P)$ given by

$$\Phi(z)=(P+zP^\perp)U^*\quad\text{and}\quad\Psi(z)=U(P^\perp+zP)$$

for all $z\in\mathbb{D}$.

In particular, $\mathcal{Q}=ran\Pi$ is a joint $(M_\Phi^*,M_\Psi^*,M_z^*)$-invariant subspace of $A^2_m(\mathcal{E})$ such that

$$T_1\cong P_\mathcal{Q}M_\Phi|_\mathcal{Q},\ T_2\cong P_\mathcal{Q}M_\Psi|_\mathcal{Q}\ \text{and}\ T\cong P_\mathcal{Q}M_z|_\mathcal{Q}.$$

Proof. We first consider the isometry U as in (4.8) and by adding an infinite dimensional Hilbert space $\mathcal{D}$, if necessary, we extend it to a unitary on $\mathcal{E}:=(\mathcal{D}\oplus\mathcal{D}_{m,T,T_1})\oplus\mathcal{D}_{m,T,T_2}$. We continue to denote the unitary by U, and therefore we have a unitary $U:\mathcal{E}\to\mathcal{E}$ which satisfies

$$U\big(0_\mathcal{D},D_{m,T,T_1}T_2^*h,D_{m,T,T_2}h\big)=\big(0_\mathcal{D},D_{m,T,T_1}h,D_{m,T,T_2}T_1^*h\big)\quad(h\in\mathcal{H}).$$

Also we view the isometry V in (4.7), as an isometry $V:\mathcal{D}_{m,T}\to\mathcal{E}$ defined by

$$V(D_{m,T}h)=(0_\mathcal{D},D_{m,T,T_1}h,D_{m,T,T_2}T_1^*h)\quad(h\in\mathcal{H}).$$

Since T is a pure m-hypercontraction, then the canonical dilation map $\Pi_{m,T}:\mathcal{H}\to A^2_m(\mathcal{D}_{m,T})$ is an isometry, and as a result

$$\Pi_V=(I\otimes V)\Pi_{m,T}:\mathcal{H}\to A^2_m(\mathcal{E})\tag{4.9}$$

is also an isometry. The isometry Π_V will be the dilation map in this context.

To complete the proof of the theorem, we construct unitaries which satisfy the hypothesis of Lemma 4.1. To this end, we consider the inclusion maps $\iota_1 : \mathcal{D} \oplus \mathcal{D}_{m,T,T_1} \to \mathcal{E}$ and $\iota_2 : \mathcal{D}_{m,T,T_2} \to \mathcal{E}$ defined by

$$\iota_1(h, k_1) = (h, k_1, 0) \text{ and } i_2(k_2) = (0, 0, k_2), (h \in \mathcal{D}, k_1 \in \mathcal{D}_{m,T,T_1}, k_2 \in \mathcal{D}_{m,T,T_2}).$$

We also consider the orthogonal projection $P = \iota_2\iota_2^*$. Then it is easy to check that

$$\begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} : \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,T_1}) \to \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,T_1})$$

and

$$\begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_{m,T,T_2} \to \mathcal{E} \oplus \mathcal{D}_{m,T,T_2}$$

are unitary. The unitary

$$U_1 := \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & i_1 \\ i_1^* & 0 \end{bmatrix} = \begin{bmatrix} UP & Ui_1 \\ i_1^* & 0 \end{bmatrix} : \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,T_1}) \to \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,T_1}),$$

satisfies

$$\begin{aligned} U_1 \begin{bmatrix} VD_{m,T}h \\ D_{m,T,T_1}T^*h \end{bmatrix} &= \begin{bmatrix} UP & Ui_1 \\ i_1^* & 0 \end{bmatrix} \begin{bmatrix} VD_{m,T}h \\ D_{m,T,T_1}T^*h \end{bmatrix} \\ &= \begin{bmatrix} U(0_{\mathcal{D}}, D_{m,T,T_1}T_2^*T_1^*h, D_{m,T,T_2}T_1^*h) \\ (0_{\mathcal{D}}, D_{m,T,T_1}h) \end{bmatrix} \\ &= \begin{bmatrix} (0_{\mathcal{D}}, D_{m,T,T_1}T_1^*h, D_{m,T,T_2}T_1^{*2}h) \\ (0_{\mathcal{D}}, D_{m,T,T_1}h) \end{bmatrix} \\ &= \begin{bmatrix} VD_{m,T}T_1^*h \\ (0_{\mathcal{D}}, D_{m,T,T_1}h) \end{bmatrix}, \end{aligned}$$

for all $h \in \mathcal{H}$. Subsequently, a similar computation also shows that the unitary

$$U_2 := \begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_{m,T,T_2} \to \mathcal{E} \oplus \mathcal{D}_{m,T,T_2},$$

satisfies

$$U_2(V\mathcal{D}_{m,T}h, D_{m,T,T_2}T^*h) = (VD_{m,T}T_2^*h, D_{m,T,T_2}h),$$

for all $h \in \mathcal{H}$. The proof now follows by appealing Lemma 4.1 to the unitaries U_1 and U_2. ■

Remark 4.3. *The converse of the above theorem is also true. That is, if (T_1, T_2, T) is a triple of commuting contractions on $\mathcal{H}$ and if (T_1, T_2, T) dilates to (M_Φ, M_Ψ, M_z) on $A^2_m(\mathcal{E})$ for some Hilbert space $\mathcal{E}$ where Φ and Ψ are $\mathcal{B}(\mathcal{E})$-valued canonical Schur functions on $\mathbb{D}$ corresponding to a triple $(\mathcal{E}, U, P)$, then $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$ and $T = T_1T_2$. This follows immediately from the fact that $(M_\Phi, M_\Psi) \in \mathcal{F}_m(A^2_m(\mathcal{E}))$ and $M_\Phi M_\Psi = M_\Psi M_\Phi = M_z$.*

Having obtained explicit dilations for the pure case, we now drop the pure assumption and find dilations for the general case.

THEOREM 4.4. *Let $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$ with $T = T_1T_2$. Then there exist a triple $(\mathcal{E}, U, P)$ consists of a Hilbert space $\mathcal{E}$, a unitary U and an orthogonal projection P in $\mathcal{B}(\mathcal{H})$, a Hilbert space $\mathcal{R}$, a pair of commuting unitaries (W_1, W_2) on a Hilbert space $\mathcal{R}$ with $W = W_1W_2$ and an isometry $\Pi : \mathcal{H} \to A^2_m(\mathcal{E})$ such that*

$$\Pi T_1^* = (M_\Phi \oplus W_1)^*\Pi,\ \Pi T_2^* = (M_\Psi \oplus W_2)^*\Pi \text{ and } \Pi T^* = (M_z \oplus W)^*\Pi,$$

where Φ and Ψ are the $\mathcal{B}(\mathcal{E})$-valued canonical Schur function on $\mathbb{D}$ corresponding to the triple $(\mathcal{E}, U, P)$ given by

$$\Phi(z) = (P + zP^\perp)U^* \quad \text{and } \Psi(z) = U(P^\perp + zP)$$

for all $z \in \mathbb{D}$.

In particular, $\mathcal{Q} = ran\Pi$ is a joint $(M_z^* \oplus W^*, M_\Phi^* \oplus W_1^*, M_\Psi^* \oplus W_2^*)$-invariant subspace of $A^2_m(\mathcal{E}) \oplus \mathcal{R}$ such that

$$T_1 \cong P_\mathcal{Q}(M_\Phi \oplus W_1)|_\mathcal{Q}, T_2 \cong P_\mathcal{Q}(M_\Psi \oplus W_2)|_\mathcal{Q} \quad \text{and } T \cong P_\mathcal{Q}(M_z \oplus W)|_\mathcal{Q}.$$

Proof. Let $(\mathcal{E}, U, P)$ be as in Theorem 4.2, and let V be as in (4.7). Then by the same way as it is done in the proof of Theorem 4.2, we have

$$\Pi_V T_1^* = M_\Phi^*\Pi_V, \Pi_V T_2^* = M_\Psi^*\Pi_V \quad \text{and } \Pi_V T^* = M_z^*\Pi_V, \tag{4.10}$$

where $\Phi(z) = (P + zP^\perp)U^*$ and $\Psi(z) = U(P^\perp + zP)$ for all $z \in \mathbb{D}$, $\Pi_V = (I \otimes V)\Pi_{m,T}$ and $\Pi_{m,T} : \mathcal{H} \to A^2_m(\mathcal{D}_{m,T}), h \mapsto D_{m,T}(I - zT^*)^{-m}h$ is the canonical dilation map. However, note that Π_V is not an isometry in general. To make it an isometry we follow the construction of the dilation map as in the proof of Theorem 2.4.

Let $Q_{m,T}$ be the positive operator defined in (2.3) by taking strong operator limit of the decreasing sequence of positive operators $\{f_r^{(m)}(T, T^*)\}_{r=0}^\infty$ where

$$f_r^{(m)}(T, T^*) = 1 - \sum_{k=0}^{r-1} w_{m,k} T^k K_m^{-1}(T, T^*) T^{*k} \quad (r \geq 0).$$

It also follows from the proof of Theorem 2.4 that

$$TQ_{m,T}^2T^* = Q_{m,T}^2.$$

We claim here that $Q_{m,T}^2 \geq T_iQ_{m,T}T_i^*$ for $i = 1, 2$. We prove the inequality for $i = 1$ as the proof is similar for $i = 2$. To this end, it is enough to show that

$$f_r^{(m)}(T, T^*) - T_1 f_r^{(m)}(T, T^*)T_1^* \geq 0,$$

for all $r \geq 0$. For a fixed $r \geq 0$, we use induction on m. Since $f_r^{(1)}(T, T^*) = T^rT^{*r}$, it is easy to see that the inequality holds for $m = 1$. We assume that for some $1 \leq n < m$,

$$f_r^{(n)}(T, T^*) - T_1 f_r^{(n)}(T, T^*)T_1^* \geq 0.$$

Now observe that

$$\begin{aligned}
&f_r^{(n+1)}(T,T^*)\\
&= 1-\sum_{k=0}^{r-1} w_{n+1,k}T^k(K_n^{-1}(T,T^*)-TK_n^{-1}(T,T^*)T^*)T^{*k}\\
&= 1-K_n^{-1}(T,T^*)-\sum_{k=1}^{r-1}(w_{n+1,k}-w_{n+1,k-1})T^kK_n^{-1}(T,T^*)T^{*k}+w_{n+1,r-1}T^rB_n^{-1}(T,T^*)T^{*r}\\
&= \big(1-\sum_{k=0}^{r-1} w_{n,k}T^kK_n^{-1}(T,T^*)T^{*k}\big)+w_{n+1,r-1}T^rK_n^{-1}(T,T^*)T^{*r}\\
&= f_r^{(n)}(T,T^*)+w_{n+1,r-1}T^rK_n^{-1}(T,T^*)T^{*r}.
\end{aligned}$$

Then this implies

$$\begin{aligned}
&f_r^{(n+1)}(T,T^*)-T_1f_r^{(n+1)}(T,T^*)T_1^*\\
&= \big(f_r^{(n)}(T,T^*)-T_1f_r^{(n)}(T,T^*)T_1^*\big)+w_{n+1,r-1}T^r\big(K_n^{-1}(T,T^*)-T_1K_n^{-1}(T,T^*)T_1^*\big)T^{*n}\geq 0.
\end{aligned}$$

Here we have used the fact that $K_n^{-1}(T,T^*)-T_1K_n^{-1}(T,T^*)T_1^*\geq 0$ for $n=1,\dots,m$. This establishes our claim and therefore, $Q_{m,T}^2\geq T_iQ_{m,T}^2T_i^*$ for $i=1,2$. Then by Lemma 2.3, there exists a contraction X_i on $\mathcal{Q}_{m,T}$ such that

$$X_i^*Q_{m,T}=Q_{m,T}T_i^* \quad (i=1,2). \tag{4.11}$$

Further, since $Q_{m,T}^2=TQ_{m,T}^2T^*$, there is an isometry X^* on $\mathcal{Q}_{m,T}$ such that $X^*Q_{m,T}=Q_{m,T}T^*$. It is now evident that $X^*=X_1^*X_2^*=X_2^*X_1^*$, and therefore X_i^* is also an isometry for $i=1,2$. Let (W_1^*,W_2^*,W^*) on $\mathcal{R}\supset\mathcal{Q}_{m,T}$ be the minimal unitary extension of (X_1^*,X_2^*,X^*) with $W^*=W_1^*W_2^*$.

Following Theorem 2.4, consider the map $\Pi:\mathcal{H}\to A_m^2(\mathcal{E})\oplus\mathcal{R}$ defined by

$$\Pi(h)=(\Pi_Vh,Q_{m,T}h),\quad (h\in\mathcal{H}).$$

Then, by Proposition 2.2 and the fact that V is an isometry, it follows that Π is an isometry. Moreover, it follows from the relations (4.10) and (4.11) that

$$\Pi T_1^*=(M_\Phi^*\oplus W_1^*)\Pi, \Pi T_2^*=(M_\Psi^*\oplus W_2^*)\Pi \;\text{ and }\; \Pi T^*=(M_z^*\oplus W^*)\Pi.$$

This completes the proof of the theorem.

■

We conclude the section with a remark which is similar to the pure case.

Remark 4.5. *The converse of the above theorem is also true. This follows from the fact that* $(M_\Phi\oplus W_1, M_\Psi\oplus W_2)\in\mathcal{F}_m(A_m^2(\mathcal{E})\oplus\mathcal{R})$.

5. Factorization of hypercontractions

Combining the dilation results, Theorem 4.2 and Theorem 4.4, obtained in the previous section with Remark 4.3 and Remark 4.5, we get the following immediate characterization of factors in the class $\mathcal{F}_m(\mathcal{H})$.

Theorem 5.1. *Let (T_1, T_2) be a pair of commuting contractions on $\mathcal{H}$. Then the following are equivalent:*

(i) $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$*;*

(ii) *there exist a pair of commuting unitaries (W_1, W_2) on a Hilbert space $\mathcal{R}$ with $W = W_1W_2$ and $\mathcal{B}(\mathcal{E})$-valued canonical Schur functions*

$$\Phi(z) = (P + zP^{\perp})U^* \quad \text{and } \Psi(z) = U(P^{\perp} + zP), \quad (z \in \mathbb{D})$$

corresponding to a triple $(\mathcal{E}, U, P)$ consisting of a Hilbert space $\mathcal{E}$, a unitary U and an orthogonal projection P in $\mathcal{B}(\mathcal{E})$ such that $\mathcal{Q}$ is a joint $(M_z^* \oplus W^*, M_\Phi^* \oplus W_1^*, M_\Psi^* \oplus W_2^*)$-invariant subspace of $A_m^2(\mathcal{E}) \oplus \mathcal{R}$,

$$T_1 \cong P_{\mathcal{Q}}(M_\Phi \oplus W_1)|_{\mathcal{Q}}, T_2 \cong P_{\mathcal{Q}}(M_\Psi \oplus W_2)|_{\mathcal{Q}}, \text{ and } T \cong P_{\mathcal{Q}}(M_z \oplus W)|_{\mathcal{Q}}.$$

In particular, if T_1T_2 is a pure contraction, then the Hilbert space $\mathcal{R} = \{0\}$.

It is now clear that the above theorem is obtained by realizing a factor $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$ on the dilation space $A_m^2(\mathcal{E}) \oplus \mathcal{R}$ of $T = T_1T_2$. However, one would expect to realize (T_1, T_2) on the canonical dilation space of T as in Theorem 2.4.

To this end, we first consider $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$ with $T = T_1T_2$ is a pure contraction. Let Π_V be the dilation map as in Theorem 4.2, that is

$$\Pi_V T_1^* = M_\Phi^* \Pi_V, \Pi_V T_2^* = M_\Psi^* \Pi_V \quad \text{and } \Pi_V T^* = M_z^* \Pi_V$$

and, by (4.9),

$$\Pi_V = (I \otimes V)\Pi_{m,T},$$

where $\Pi_{m,T}$ is the isometric canonical dilation map corresponding to the pure m-hypercontraction T and $V : \mathcal{D}_{m,T} \to \mathcal{E}$ is an isometry. Then, by the definition of Π_V, the above intertwining relations yield

$$\Pi_{m,T} T_1^* = (I \otimes V^*) M_\Phi^* (I \otimes V)\Pi_{m,T} \quad \text{and } \Pi_{m,T} T_2^* = (I \otimes V^*) M_\Psi^* (I \otimes V)\Pi_{m,T}.$$

Set

$$\tilde{\Phi}(z) := V^*\Phi(z)V \quad \text{and } \tilde{\Psi}(z) := V^*\Psi(z)V, \quad (z \in \mathbb{D}).$$

Then $\tilde{\Phi}$ and $\tilde{\Psi}$ are $\mathcal{B}(\mathcal{D}_{m,T})$-valued Schur functions on $\mathbb{D}$ such that

$$\Pi_{m,T} T_1^* = M_{\tilde{\Phi}}^* \Pi_{m,T}, \Pi_{m,T} T_2^* = M_{\tilde{\Psi}}^* \Pi_{m,T}.$$

Observant reader may have noticed that $\tilde{\Phi}$ and $\tilde{\Psi}$ do not commute, in general. However, $P_{\mathcal{Q}} M_{\tilde{\Phi}}|_{\mathcal{Q}}$ and $P_{\mathcal{Q}} M_{\tilde{\Psi}}|_{\mathcal{Q}}$ commute with each other and

$$P_{\mathcal{Q}} M_z|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\tilde{\Phi}} M_{\tilde{\Psi}}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{\tilde{\Psi}} M_{\tilde{\Phi}}|_{\mathcal{Q}},$$

where $\mathcal{Q} = \text{ran}\Pi_{m,T}$. Thus we have proved the following:

Theorem 5.2. *Let T be a pure m-hypercontraction on $\mathcal{H}$. Then the following are equivalent.*

(*i*) $T = T_1T_2$ *for some* $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$;

(*ii*) *there exist* $\mathcal{B}(\mathcal{D}_{m,T})$*-valued Schur functions*

$$\tilde{\Phi}(z) = V^*(P + zP^{\perp})U^*V, \quad \text{and } \tilde{\Psi}(z) = V^*U(P^{\perp} + zP)V \quad (z \in \mathbb{D})$$

for some Hilbert space $\mathcal{E}$, *isometry* $V : \mathcal{D}_{m,T} \to \mathcal{E}$, *unitary* $U : \mathcal{E} \to \mathcal{E}$ *and projection* P *in* $\mathcal{B}(\mathcal{E})$ *such that* $\mathcal{Q}$ *is a joint* $(M^*_{\tilde{\Phi}}, M^*_{\tilde{\Psi}})$*-invariant subspace of* $A^2_m(\mathcal{D}_{m,T})$,

$$P_{\mathcal{Q}}M_z|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\tilde{\Phi}\tilde{\Psi}}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\tilde{\Psi}\tilde{\Phi}}|_{\mathcal{Q}},$$

and

$$T_1 \cong P_{\mathcal{Q}}M_{\tilde{\Phi}}|_{\mathcal{Q}}, \text{ and } T_2 \cong P_{\mathcal{Q}}M_{\tilde{\Psi}}|_{\mathcal{Q}}.$$

We also have the following analogous result for general m-hypercontractions.

Theorem 5.3. *Let* T *be an* m*-hypercontraction on* $\mathcal{H}$*. Then the following are equivalent.*

(*i*) $T = T_1T_2$ *for some* $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$;

(*ii*) *there exist a commuting pair of unitaries* (W_1, W_2) *on a Hilbert space* $\mathcal{R}$ *with* $W = W_1W_2$ *and* $\mathcal{B}(\mathcal{D}_{m,T})$*-valued Schur functions*

$$\tilde{\Phi}(z) = V^*(P + zP^{\perp})U^*V, \quad \text{and } \tilde{\Psi}(z) = V^*U(P^{\perp} + zP)V \quad (z \in \mathbb{D})$$

for some Hilbert space $\mathcal{E}$, *isometry* $V : \mathcal{D}_{m,T} \to \mathcal{E}$, *unitary* $U : \mathcal{E} \to \mathcal{E}$ *and projection* P *in* $\mathcal{B}(\mathcal{E})$ *such that* $\mathcal{Q}$ *is a joint* $(M^*_{\tilde{\Phi}} \oplus W^*_1, M^*_{\tilde{\Psi}} \oplus W^*_2)$*-invariant subspace of* $A^2_m(\mathcal{D}_{m,T}) \oplus \mathcal{R}$,

$$P_{\mathcal{Q}}(M_z \oplus W)|_{\mathcal{Q}} = P_{\mathcal{Q}}(M_{\tilde{\Phi}\tilde{\Psi}} \oplus W)|_{\mathcal{Q}} = P_{\mathcal{Q}}(M_{\tilde{\Psi}\tilde{\Phi}} \oplus W)|_{\mathcal{Q}},$$

and

$$T_1 \cong P_{\mathcal{Q}}(M_{\tilde{\Phi}} \oplus W_1)|_{\mathcal{Q}}, \text{ and } T_2 \cong P_{\mathcal{Q}}(M_{\tilde{\Psi}} \oplus W_2)|_{\mathcal{Q}}.$$

An immediate consequence of the above results is a similar factorization result for subnormal operators. Recall that an operator is subnormal if it has a normal extension. A well-known characterization of subnormal operators due to Agler is the following: a contraction T on a Hilbert space $\mathcal{H}$ is subnormal if and only if T is an m-hypercontraction for all $m \in \mathbb{N}$ (see [2]). We set

$$\mathcal{F}_\infty(\mathcal{H}) := \bigcap_m \mathcal{F}_m(\mathcal{H}).$$

By the above characterization, if $(T_1, T_2) \in \mathcal{F}_\infty(\mathcal{H})$ then $T = T_1T_2$ is a subnormal operator. Thus $\mathcal{F}_\infty(\mathcal{H})$ contains contractive factors of subnormal operators on $\mathcal{H}$. A characterization of $\mathcal{F}_\infty(\mathcal{H})$ is in order.

Theorem 5.4. *Let* T *be a subnormal operator on* $\mathcal{H}$*. Then the following are equivalent.*

(*i*) $T = T_1T_2$ *for some* $(T_1, T_2) \in \mathcal{F}_\infty(\mathcal{H})$;

(*ii*) *for each* $m \in \mathbb{N}$, *there exist a commuting pair of unitaries* $(W_{1,m}, W_{2,m})$ *on a Hilbert space* $\mathcal{R}_m$ *with* $W_m = W_{1,m}W_{2,m}$ *and* $\mathcal{B}(\mathcal{D}_{m,T})$*-valued Schur functions*

$$\tilde{\Phi}_m(z) = V^*_m(P_m + zP^{\perp}_m)U^*_mV_m, \quad \text{and } \tilde{\Psi}_m(z) = V^*_mU_m(P^{\perp}_m + zP_m)V_m \quad (z \in \mathbb{D})$$

for some Hilbert space $\mathcal{E}_m$, isometry $V_m : \mathcal{D}_{m,T} \to \mathcal{E}_m$, unitary $U_m : \mathcal{E}_m \to \mathcal{E}_m$ and projection P_m in $\mathcal{B}(\mathcal{E}_m)$ such that $\mathcal{Q}_m$ is a joint $(M^*_{\tilde{\Phi}_m} \oplus W^*_{1,m}, M^*_{\tilde{\Psi}_m} \oplus W^*_{2,m})$-invariant subspace of $A^2_m(\mathcal{D}_{m,T}) \oplus \mathcal{R}_m$,

$$P_{\mathcal{Q}_m}(M_z \oplus W_m)|_{\mathcal{Q}_m} = P_{\mathcal{Q}_m}(M_{\tilde{\Phi}_m\tilde{\Psi}_m} \oplus W_m)|_{\mathcal{Q}_m} = P_{\mathcal{Q}_m}(M_{\tilde{\Psi}_m\tilde{\Phi}_m} \oplus W_m)|_{\mathcal{Q}_m},$$

and

$$T_1 \cong P_{\mathcal{Q}_m}(M_{\tilde{\Phi}_m} \oplus W_{1,m})|_{\mathcal{Q}_m}, \text{ and } T_2 \cong P_{\mathcal{Q}_m}(M_{\tilde{\Psi}_m} \oplus W_{2,m})|_{\mathcal{Q}_m}.$$

6. Examples and concluding remark

In this section, we find an example of a pair of commuting 2×2 contractive matrices such that their product is a 2-hypercontraction but the pair fails to belong in $\mathcal{F}_2(\mathbb{C}^2)$.

Example: For a real number r with $0 < r \leq 1$, consider a 2×2 matrix $T_r := \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$. Then by a direct calculation, it can be checked that T_r is a 2-hypercontraction if and only if $r^2 \leq \frac{1}{2}$. Also for strictly positive real numbers a and b, consider the matrix $S = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Then S is an invertible matrix and S commutes with T_r for any r. Thus, for $r \leq \frac{1}{\sqrt{2}}$, T_rS^{-1} and S are factors of the 2-hypercontraction T_r. On the other hand, again by a simple direct calculation, we have

$$K_1^{-1}(T_r, T_r^*) - SK_1^{-1}(T_r, T_r^*)S^* = \begin{bmatrix} (1-r^2)(1-a^2) - b^2 & -ab \\ -ab & 1-a^2 \end{bmatrix}. \tag{6.12}$$

Also note that S is a contraction if and only if $b \leq 1 - a^2$. So for the particular choice $r = \frac{1}{\sqrt{2}}$, $a = \frac{1}{\sqrt{2}}$ and $b = \frac{1}{2}$, we see that T_r is a 2-hypercontraction, S and T_rS^{-1} are contractions and

$$K_1^{-1}(T_r, T_r^*) - SK_1^{-1}(T_r, T_r^*)S^* = \begin{bmatrix} 0 & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

is not a positive matrix. Therefore for such a particular choice, the contractions T_rS^{-1} and S are factors of the 2-hypercontraction T_r but $(T_rS^{-1}, S) \notin \mathcal{F}_2(\mathbb{C}^2)$.

The above example shows that $\mathcal{F}_m(\mathcal{H})$ does not contain all the contractive factors of m-hypercontractions on $\mathcal{H}$ and the present article characterise a subclass of contractive factors of m-hypercontractions, namely $\mathcal{F}_m(\mathcal{H})$. We conclude the paper with the following natural question: How to characterize all the contractive factors of m-hypercontractions?

Acknowledgement: The authors are grateful to Prof. Jaydeb Sarkar for his inspiration and valuable comments. The research of the first name author is supported by the institute post doctoral fellowship of IIT Bombay. The research of the second named author is supported by DST-INSPIRE Faculty Fellowship No. DST/INSPIRE/04/2015/001094.

References

1. J. Agler, *The Arveson extension theorem and coanalytic models*, Integral Equations Operator Theory 5 (1982), 608-631.
2. J. Agler, *Hypercontractions and subnormality*, J. Operator Theory 13 (1985), 203-217.

3. C. Ambrozie, M. Englis and V. Muller, *Operator tuples and analytic models over general domains in* $\mathbb{C}^n$, J. Operator Theory 47 (2002), no. 2, 287-302.
4. S. Barik, B.K. Das, K. Haria and J. Sarkar, *Isometric dilation and von Neumann inequality for a class of tuples in the polydisc*, Tran. Amer. Math. Soc. 372 (2019), 1429-1450.
5. H. Bercovici, R.G. Douglas and C. Foias, *Canonical models for bi-isometries*, A panorama of modern operator theory and related topics, 177-205, Oper. Theory Adv. Appl., 218, Birkhauser/Springer Basel AG, Basel, 2012.
6. H. Bercovici, R.G. Douglas and C. Foias, *Bi-isometries and commutant lifting*, Characteristic functions, scattering functions and transfer functions, 51-76, Oper. Theory Adv. Appl., 197, Birkhauser Verlag, Basel, 2010.
7. H. Bercovici, R.G. Douglas and C. Foias, *On the classification of multi-isometries*, Acta Sci. Math.(Szeged) 72 (2006), 639-661.
8. C.A. Berger, L.A. Coburn and A. Lebow, *Representation and index theory for* C^**-algebras generated by commuting isometries*, J. Funct. Anal. 27 (1978), 51-99.
9. R.E. Curto and F.H. Vasilescu, *Standard operator models in the polydisc*, Indiana Univ. Math. J. 42 (1993), no. 3, 791-810.
10. R.E. Curto and F.H. Vasilescu, *Standard operator models in the polydisc, II*, Indiana Univ. Math. J. 44 (1995), no. 3, 727-746.
11. B.K. Das, R. Debnath and J. Sarkar, *On certain commuting isometries, joint invariant subspaces and* C^**-algebras*, arXiv:1711.00769.
12. B.K. Das and J. Sarkar, *Ando dilations, von Neumann inequality, and distinguished varieties*, J. Funct. Anal. 272 (2017), 2114-2131.
13. B.K. Das, S. Sarkar and J. Sarkar, *Factorization of Contraction*, Advances in Mathematics 322 (2017), 186-200.
14. R. G. Douglas, *Structure theory for operators. I.*, J. Reine Angew. Math. 232 (1968), 180-193.
15. R.G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. 17 (1996), 413-415.
16. K. Guo and R. Yang, *The core function of submodules over the bidisk*, Indiana Univ. Math. J. 53 (2004), 205-222.
17. W. He, Y. Qin and R. Yang, *Numerical invariants for commuting isometric pairs*, Indiana Univ. Math. J. 64 (2015), 1–19.
18. V. Muller and F.-H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. 117 (1993), 979–989.
19. B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, Amsterdam, 1970.
20. A. Olofsson, *Parts of adjoint weighted shifts*, J. Operator Theory 74 (2015), no. 2, 249-280.
21. Haripada Sau, *And dilations for a pair of commuting contractions: two explicit constructions and functional models*, arxiv:1710.11368.

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, 400076, India

Email address: mono@math.iitb.ac.in, monojit.hcu@gmail.com

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, 400076, India

Email address: dasb@math.iitb.ac.in, bata436@gmail.com